

5/MTH-301 Syllabus-2023

2025

(Nov-Dec)

FYUP : 5th Semester Examination

MAJOR

MATHEMATICS

(Number Theory and Ring Theory)

MTH-301

Marks : 75

Time : 3 hours

*The figures in the margin indicate full marks
for the questions*

Answer **four** questions, taking **one** from each Unit

UNIT—I

1. (a) State whether the following statements are true or false with brief justification (any four) (a, b, c, d denote positive integers) :
 $2\frac{1}{2} \times 4 = 10$
- (i) If p is a prime and $a^2 \equiv b^2 \pmod{p}$, then $p \mid a - b$.

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(Turn Over)

(2)

(ii) If a is odd, then $(a+3, a-3) = 6$.

(iii) $\exists a$ such that $5a^2 + 22$ is divisible by 17.

(iv) If $\frac{p-1}{2} \equiv -1 \pmod{p}$, then p is a prime.

(v) If $a^2 \mid c^3$, then $a \mid c$.

(vi) If $b \mid a^2 + 1$, then $b \mid a^4 + 1$.

(vii) The set of integers
 $\{1, 2, 4, 8, 7, -2, -4, -8\}$
forms a reduced residue system modulo 15.

(b) State and prove Wilson's theorem. 5

(c) Find the last two digits in ordinary decimal representation of 7^{200} . 3

2. (a) Prove that $n^{12} - a^{12}$ is divisible by 91 if n and a are prime to 91. 3

(b) Show that $3, 6, 9, \dots, 3m$ is a complete residue system mod m , if $3 \times m$. 3

(c) If q is a prime factor of $a^2 + b^2$ and if $q \equiv 3 \pmod{4}$, show that $q \mid a$ and $q \mid b$. 4

(3)

(d) If p is an odd prime, show that

$$1^2 \cdot 3^2 \cdot 5^2 \dots (p-2)^2 \equiv (-1)^{\frac{p+1}{2}} \pmod{p} \quad 4$$

(e) Prove that $11 \mid 2^{4n+3} + 5^{n+2}$ for every positive integer n . 4

UNIT—II

3. (a) Find all integers that give remainders 1, 2, 3 when divided by 4, 5 and 7 respectively. 5

(b) Solve the congruence $15x \equiv 25 \pmod{35}$. 4

(c) Show that $\mu(n)$ is multiplicative and

$$\sum_{d \mid n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases} \quad 4$$

(d) If k is the number of distinct prime factors of n (a positive integer), show that

$$\sum_{d \mid n} |\mu(d)| = 2^k \quad 4$$

(e) Give an example to show that if $f(n)$ is totally multiplicative, $F(n)$ need not be totally multiplicative, when

$$F(n) = \sum_{d \mid n} f(d) \quad 2$$

(4)

4. (a) What is the highest power of 6 dividing $\underline{523}$? 3

(b) If $\underline{96}$ were written out in the ordinary decimal notation without factorial sign, find the number of zeros that would be at the right end. 3

(c) For any two real numbers x and y , show that

$$[x] + [y] \leq [x + y] \leq [x] + [y] + 1 \quad 4$$

(d) Prove that

$$\prod_{d|n} d = n^{\frac{d(n)}{2}} \quad 3$$

(e) If a and b are positive integers such that $(a, b) = 1$ and c is a real number such that ac, bc are integers, prove that c is an integer. Hence show that

$$c = \frac{\underline{n}}{\underline{a} \ \underline{b}}$$

is an integer if $(a, b) = 1$ and $a + b = n + 1$. 6

UNIT—III

5. (a) Let $a \in R$, a ring. Show that $\{x \in R : ax = 0\}$ is a subring of R . 4

(b) Show that a finite integral domain is a field. 5

(5)

- (c) Let A be an ideal of a commutative ring $R \neq \{0\}$. Show that R/A is an integral domain if and only if A is a prime ideal. 5
- (d) Let $J = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$. Show that J is a commutative ring under addition and multiplication of real numbers. Is it a field? Justify. 5
6. (a) If U and V are two ideals of R , show that
 $U + V = \{x + y : x \in U \text{ and } y \in V\}$
is an ideal of R . 3
- (b) In a ring R , if $x^2 = x \forall x \in R$, show that R is commutative. 5
- (c) Show that the units of the ring
 $\mathbb{Z}[i] = \{a + ib : a, b \in \mathbb{Z}\}$
are $+1$ and -1 . 4
- (d) Let R be a commutative ring with unity whose only ideals are (0) and R itself. Show that R is a field. 5
- (e) In the ring of integers $\mathbb{Z}$, find a positive integer m s.t. $2\mathbb{Z} + 3\mathbb{Z} = m\mathbb{Z}$. 2

UNIT—IV

7. (a) Define a ring homomorphism. Show that the map $\phi: \mathbb{Z}_4 \rightarrow \mathbb{Z}_{10}$ defined by $\phi(x) = 5x$ is a ring homomorphism. $1+3=4$

(6)

(b) If $\phi: R \rightarrow S$ is a ring homomorphism, show that—

(i) $\phi(A)$ is an ideal of S whenever A is an ideal of R ;

(ii) $\ker \phi$ is an ideal of R ;

(iii) $\phi^{-1}(B)$ is an ideal of R for every ideal B of S ;

(iv) R is commutative implies $\phi(R)$ is commutative. 2×4=8

(c) Show that the only ring automorphism of the set of rational numbers is the identity map. 4

(d) Show that the map $\phi: R \rightarrow \mathbb{Z}$ defined by

$$\begin{bmatrix} 1 & b \\ 0 & c \end{bmatrix} \mapsto a$$

where

$$R = \left\{ \begin{bmatrix} 1 & b \\ 0 & c \end{bmatrix} : a, b, c \in \mathbb{Z} \right\}$$

is a ring homomorphism. 3

8. (a) Show that

$$\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\}$$

is an integral domain but not a unique factorization domain. 5

(7)

- (b) In a PID, show that an element is irreducible if and only if it is prime. 4
- (c) In the integral domain $\mathbb{Z}[\sqrt{-3}]$, give an example of an irreducible element which is not a prime. 3
- (d) Give an example of a prime ideal which is not maximal. 3
- (e) In a finite commutative ring, show that every prime ideal is maximal. 4
