

3/MTH-201 Syllabus-2023

2 0 2 5

(Nov-Dec)

FYUP : 3rd Semester Examination

MAJOR

MATHEMATICS

(Group Theory)

MTH-201

Marks : 75

Time : 3 hours

*The figures in the margin indicate full marks
for the questions*

Answer **four** questions, taking **one** from each Unit

UNIT—I

1. (a) Let $G = \{A \in M_2(\mathbb{R}) \mid \det(A) = 1\}$. Prove that G forms a non-Abelian group under matrix multiplication. 5
- (b) Explain why $\mathbb{Z}_8$ does not form a group under multiplication modulo 8. 2
- (c) Let $\mathbb{C}^* = \{a+ib \mid a, b \in \mathbb{R}, a^2 + b^2 \neq 0\}$ be the group of non-zero complex numbers under multiplication. Find the inverse of an element $a+ib$ in $\mathbb{C}^*$. 3

- (d) Prove that G is an Abelian group if and only if $(ab)^2 = a^2b^2$ for all $a, b \in G$. 4
- (e) Prove that if every element of a group G is its own inverse, then G is Abelian. 4
2. (a) Prove that the set of all positive rational numbers forms an Abelian group under the operation defined by $a * b = \frac{ab}{2}$. 4
- (b) Let G be a group, and $a, b, c \in G$. Prove that $ab = ac$ implies $b = c$. 2
- (c) Determine the order of every element of $\mathbb{Z}_6$ under addition modulo 6. 3
- (d) Let G be a group and $a \in G$ such that $\text{ord}(a) = n$. If $a^k = e$ for some integer k , then prove that n divides k . 3
- (e) Give an example of an infinite group containing an element of finite order. 2
- (f) Prove that in any group, an element and its inverse have the same order. 4

UNIT—II

3. (a) Let S_n denote the set of all permutations on the set $A = \{1, 2, 3, \dots, n\}$, where $n \geq 3$. Prove that—
- (i) S_n forms a group under composition of maps; 3+2=5
- (ii) S_n is non-Abelian.

(3)

(b) Find all subgroups of S_3 . 3

(c) What is the inverse of a k -cycle in S_n ?
Prove that the order of a k -cycle is k .
3+3=6

(d) Let $\alpha = (12345)(678)$ and $\beta = (23847)(56)$
be permutations in S_8 . Find $\alpha^2\beta$ and
 $\beta^{-1}\alpha$. 5

4. (a) Prove that disjoint cycles in S_n
commute. 4

(b) Prove that every permutation in S_n
can be expressed as a product of
transpositions. Is the representation, of
a permutation as a product of
transpositions, unique? Justify your
answer. 4+1=5

(c) Determine whether the following
permutations are odd or even : 3

(i) $(12)(345)(789)$

(ii) $(62351)(7249)$

(d) Prove that A_n , the set of even
permutations in S_n , is a subgroup of
 S_n . 4

(e) Show that the product of any two odd
permutations is an even permutation. 3

UNIT—III

5. (a) Prove that every cyclic group is Abelian. 3
- (b) If a is a generator of a group G , prove that a^{-1} is also a generator of G . 4
- (c) Let G be a group and $a \in G$. Define the centralizer of a in G , denoted by $C(a)$. Prove that $C(a)$ is a subgroup of G . 1+3=4
- (d) Prove that every subgroup of $\mathbb{Z}$ is of the form $n\mathbb{Z}$, for some integer n . 4
- (e) Find all proper subgroups of the quaternion group
- $$Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$$
- Are all of them cyclic? 4
6. (a) Let H be a subgroup of G and $a, b \in G$. Prove that $aH = bH$ if and only if $a^{-1}b \in H$. 4
- (b) Determine the index of $3\mathbb{Z}$ in $\mathbb{Z}$. 3
- (c) Let G be a finite group and $a \in G$. Prove that $\text{ord}(a) = |\langle a \rangle|$, and further show that $\text{ord}(a)$ divides $|G|$. 5
- (d) Prove that an infinite cyclic group has exactly two generators. 3
- (e) State and prove Fermat's Little theorem. 4

UNIT—IV

7. (a) Prove that the intersection of any two normal subgroups of G is normal in G . 4
- (b) If H is a subgroup of index 2 in G , prove that H is normal in G . Explain why A_3 is a normal subgroup of S_3 . 4+2=6
- (c) Determine, up to isomorphism, all quotient groups of $\mathbb{Z}_{12}$. 5
- (d) Let $\phi: G \rightarrow G'$ be a group homomorphism. Prove that ϕ is one-one if and only if $\ker \phi = \{e\}$. 4
8. (a) State and prove the fundamental theorem of group homomorphism. 1+5=6
- (b) Let G be a group and $a \in G$. Define a map $f_a: G \rightarrow G$ as $f_a(x) = a^{-1}xa$ for all $x \in G$. Prove that f_a is an automorphism. 4
- (c) Find all automorphisms of $\mathbb{Z}_6$. 4
- (d) Prove that any non-cyclic group of order 4 is isomorphic to the Klein 4-group. 5
