

3/MTH-201 Syllabus-2023

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(December)

FYUP : 3rd Semester Examination

MAJOR

MATHEMATICS

(Group Theory)

MTH-201

Marks : 75

Time : 3 hours

*The figures in the margin indicate full marks
for the questions*

UNIT—1

1. (a) Let G be the set of all real 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $ad - bc \neq 0$ is a rational number. Prove that G forms a group under matrix multiplication. 5

- (b) Let G be a group. Show that for all $a, b \in G$, $(a^{-1})^{-1} = a$ and $(a \cdot b)^{-1} = b^{-1} \cdot a^{-1}$.

2+3=5

D25/981

(Turn Over)

- (c) Let G be a group, $a \in G$ and $b \in G$. Let n be a positive integer. Show that $(aba^{-1})^n = aba^{-1}$, if and only if, $b = b^n$. 4
- (d) If G is a finite group, then show that there exists a positive integer N such that $a^N = e$ for all $a \in G$. 4
2. (a) If G is a group of even order, then prove that it has an element $a \neq e$ satisfying $a^2 = e$. 4
- (b) Show that the set $\{1, -1, i, -i\}$ forms a group with respect to multiplication. Is this group an abelian group? Justify. 5
- (c) Let G be a group and $a, b \in G$. Show that the equations $a \cdot x = b$ and $y \cdot a = b$ have unique solutions for x and y in G . 4
- (d) Let G be a group. Show that the identity element of G is unique. Also, show that every $a \in G$ has a unique inverse in G . 2+3=5

UNIT—2

3. (a) Show that every permutation can be expressed as a product of disjoint cycles. 6
- (b) Express $(1, 2, 3) (4, 5) (1, 6, 7, 8, 9) (1, 5)$ as the product of disjoint 2 cycles. 3

- (c) Let A_n be the set of even permutations in S_n . Show that A_n forms a group of order $\frac{n!}{2}$. 5
- (d) Let $G = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$. Define a product in G by $a^4 = e, b^2 = a^2, ba = a^{-1}b$. Show that G forms a group (called the quaternion group). 5
4. (a) Determine which of the following permutations are even : 3
- (i) $(1\ 2\ 3\ 4\ 5)(1\ 2\ 3)(4\ 5)$
- (ii) $(1\ 2\ 3)(4\ 5)(1\ 6\ 7\ 8)$
- (b) Compute $a^{-1}ba$, where $a = (1\ 3\ 5)(1\ 2)$ and $b = (1\ 5\ 7\ 9)$. 3
- (c) Let $D_{2n} = \{e, a, a^2, \dots, a^{n-1}, b, ba, \dots, ba^{n-1}\}$ be a set of $2n$ elements. Define the product in D_{2n} by the relations $a^n = e, b^2 = e$ and $ab = ba^{-1}$.
- (i) Find the form of the product $(x^i y^j)(x^k y^l)$ as $x^\alpha y^\beta$. 3
- (ii) Using this, prove that G forms a group. 5
- (iii) If $n = 4$, then show that D_8 is the group of symmetries of the square. 5

UNIT—3

5. (a) State and prove Lagrange's theorem. 4
- (b) Let H be a subgroup of a group G , $a \in G$ and $aHa^{-1} = \{aha^{-1} \mid h \in H\}$. Show that aHa^{-1} is a subgroup of G . Also, show that if H is finite, then $o(H) = o(aHa^{-1})$. 3+3=6
- (c) Prove that any subgroup of a cyclic group is itself a cyclic group. 4
- (d) Let $G = \langle a \rangle$ be a cyclic group of order 8 and let $H = \langle a^4 \rangle$. Find all the cosets of H in G . 3
- (e) Define the centre Z of a group G and show that Z is a subgroup of G . 2
6. (a) State and prove Euler's theorem. 4
- (b) Show that every group of prime order is cyclic. 5
- (c) Let G be a finite cyclic group of order n . Show that G has $\phi(n)$ generators. 5

- (d) If H is a subgroup of G , and $N(H) = \{a \in G \mid aHa^{-1} = H\}$, then prove that $N(H)$ is a subgroup of G and $H \subseteq N(H)$. 3+2=5

UNIT—4

7. (a) If N and M are normal subgroups of G , then prove that NM is also a normal subgroup of G . 5

- (b) If H and K are normal subgroups of a group G such that $H \subseteq K$, then show that

$$\frac{G}{K} \cong \frac{(G/H)}{(K/H)} \quad 6$$

- (c) Let G be a group and ϕ an automorphism of G . If $a \in G$ is of order $o(a) > 0$, then show that $o(\phi(a)) = o(a)$. 4

- (d) Let G be a group, N a normal subgroup of G , T an automorphism of G . Prove that $T(N)$ is a normal subgroup of G . 4

8. (a) Show that the group $I(G)$ of inner automorphisms of G is isomorphic to $\frac{G}{Z(G)}$. 5

(b) Let H and K be normal subgroups of G such that $H \cap K = \{e\}$. Show that $hk = kh$ for all $h \in H, k \in K$. 4

(c) Let G be a group, H and K subgroups of G such that K is normal in G . Then show that $H \cap K$ is normal in H and the quotient group $\frac{H}{H \cap K}$ is isomorphic to the quotient group $\frac{HK}{K}$. 1+4=5

(d) Let G be the group of non-zero complex numbers under multiplication and let N be the set of complex numbers of absolute value 1. (That is $a+bi \in N$ if $a^2 + b^2 = 1$). Show that $\frac{G}{N}$ is isomorphic to the group of all positive real numbers under multiplication. 5
