

1M/MTH-100 Syllabus-2023

2024

(December)

FYUP : 1st Semester Examination

MAJOR

MATHEMATICS

(Fundamental Mathematics—I)

MTH-100

Marks : 75

Time : 3 hours

*The figures in the margin indicate full marks
for the questions*

Answer **four** questions, choosing **one** from each Unit

UNIT—I

1. (a) Using ϵ - δ definition of continuity of a real valued function, show that $f(x) = 5x + 3$ is continuous at $x = 1$. 4

- (b) A function f is defined as

$$f(x) = \begin{cases} x+1, & \text{when } x \leq 1 \\ 3-ax^2, & \text{when } x > 1 \end{cases}$$

For what value of a , will the function be continuous at $x = 1$? 3

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(Turn Over)

(c) Evaluate

$$\lim_{x \rightarrow \infty} \sqrt{x}(\sqrt{x+4} - \sqrt{x})$$

3

(d) Does $\lim_{x \rightarrow 3} \frac{x^2 - 9}{|x - 3|}$ exist? Justify.

3

(e) Let $f: [0, 1] \rightarrow \mathbb{R}$ be a continuous function such that $f(0) > 0$ and $f(1) < 1$. Consider the function $g: [0, 1] \rightarrow \mathbb{R}$ given by $g(x) = f(x) - x \forall x \in [0, 1]$. Show that there exists a point $c \in [0, 1]$ such that $f(c) = c$.

State the theorem you have used. 4+1=5

2. (a) Determine the limit L of the function $f(x) = 3x - 11$ as $x \rightarrow 2$. Hence find δ such that for all x ,

$$0 < |x - 2| < \delta \Rightarrow |f(x) - L| < 0.003 \quad 4$$

(b) A function f is defined as

$$f(x) = \begin{cases} x^2 \cos\left(\frac{1}{x}\right), & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$$

Is $f(x)$ continuous at $x = 0$? Justify. 4

(c) State the intermediate value theorem. Use this theorem to prove that there exists a real number that is 1 less than its cube.

1+4=5

(3)

- (d) If in a certain neighbourhood of a , $f(x)$, $g(x)$, $h(x)$ be three functions such that $f(x) \leq g(x) \leq h(x)$ and

$$\lim_{x \rightarrow a} f(x) = l = \lim_{x \rightarrow a} h(x)$$

Prove that $\lim_{x \rightarrow a} g(x)$ exists and is equal to l .

5

UNIT—II

3. (a) State and prove the Lagrange's mean value theorem. 1+4=5

- (b) If $y = a \cos(\log x) + b \sin(\log x)$ for $x > 0$, prove that—

(i) $x^2 y_2 + x y_1 + y = 0$

(ii) $x^2 y_{n+2} + (2n+1) x y_{n+1} + (n^2 + 1) y_n = 0$
2+3=5

- (c) Define point of inflexion. Find the points of inflexion, if any, of the curve

$$y = \frac{1}{4} x^4 - 4x^3 + 24x^2$$
1+3=4

- (d) A function f is defined as

$$f(x) = 2x^3 + 3x^2 - 36x + 48$$

Find (i) the stationary points and (ii) the intervals on which $f(x)$ is increasing and decreasing.

1+4=5

(4)

4. (a) State Leibnitz's theorem on the product of two derivable functions of x . Apply it to find y_n if $y = x^3 e^x$. 1+3=4
- (b) Show that for $x > 0$, the curve $y = \log x$ is everywhere convex upwards but the curve $y = x \log x$ is everywhere concave upwards. 4
- (c) What is the geometrical interpretation of the derivative of a function at a point? What is the gradient of the tangent to the curve $y = x^3 - 12x + 16$ at the point $(0, 2)$? At what points on the curve are the tangents parallel to the X -axis? 1+1+2=4
- (d) State Rolle's theorem. Discuss the applicability of Rolle's theorem for the function $f(x) = \sin x \cdot \cos x$, $0 \leq x \leq \frac{\pi}{2}$. Find at least one value of x in the interval at which $f'(x) = 0$, if the theorem is applicable. 1+3=4
- (e) Using differentials, find the approximate value of $\sqrt{50}$. 3

UNIT—III

5. (a) Express $\int_a^b e^x dx$ as the limit of a sum and hence evaluate it. 4

(b) Show that

$$\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx = \frac{\pi^2}{4} \quad 4$$

(c) If $I_n = \int_0^{\pi/2} \cos^n x dx$, where n is a positive integer greater than 1, then show that

$$I_n = \frac{n-1}{n} I_{n-2} \quad 4$$

(d) Prove that the length of the circle $x^2 + y^2 = a^2$ lying between the ordinates $x = \frac{a}{2}$ and $x = a$ is $\frac{1}{3} \pi a$. 3

(e) Find the area of the surface generated by the arc $y = x^2$ between $x = 0$ and $x = \sqrt{2}$ about the Y-axis. 4

6. (a) State and prove the fundamental theorem of integral calculus. 1+4=5

(b) Find, by integration, the area in the first quadrant bounded by the ellipse

$$\frac{x^2}{16} + \frac{y^2}{9} = 1 \quad 4$$

(c) If $f(x) = |x-2| + |x-3|$, find $\int_2^4 f(x) dx$. 4

(d) Find the volume of the solid of revolution formed by the rotation of the circle $x^2 + y^2 = 4$ about the X -axis. 4

(e) Obtain the reduction formula for $\int x^n e^{ax} dx$, n being a positive integer. 2

UNIT—IV

7. (a) State the fundamental theorem of algebra. Give examples to show its failure over $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$. 1+3=4

(b) Solve $x^4 - 9x^3 + 39x^2 - 89x + 78 = 0$ if one of the roots is $2 + 3i$. 4

(c) If α, β, γ are roots of the equation $x^3 + px^2 + qx + r = 0$, find the value of $\Sigma \alpha^2 \beta$ in terms of the coefficients. 4

(d) Reduce the equation

$$x^3 - 3x^2 + 12x + 16 = 0$$

to one where the second degree term is absent and then solve it by using Cardan's method. 3+4=7

8. (a) If Z_1 and Z_2 be two complex numbers, then show that

$$|Z_1 + Z_2| \leq |Z_1| + |Z_2| \quad 3$$

(7)

- (b) Express $5x^4 - 7x^3 + 4x^2 + 2x$ as a function of $(x-3)$. 4
- (c) Find the number of real roots of the equation $x^3 + x^2 - 5x + 1 = 0$ and determine the positions of the real roots. 4
- (d) Solve
$$x^4 + x^3 - 16x^2 - 4x + 48 = 0$$
given that the product of two of its roots is 6. 5
- (e) Find the values of $(1+i)^{\frac{1}{3}}$ by using the De Moivre's theorem. 3
